

Group Analysis via Weak Symmetries For Benjamin-Bona-Mahony Equation

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Abstract

In this paper, weak symmetries of the Benjamin-Bona-Mahony (BBM) equation have been investigated. Indeed, this method has been performed by applying the non-classical symmetries of the BBM equation and the infinitesimal generators of the classical symmetry algebra of the KdV equation as the starting constraints. Similarity reduced equations as well as some exact solutions of the BBM equation are obtained via this method.

Keywords: Weak symmetry, Non-classical symmetry, Similarity reduced Equation, Benjamin-Bona-Mahony Equation.

1. Introduction

The Benjamin-Bona-Mahony equation

$$\text{BBM} : u_t + u_x + uu_x - u_{xxt} = 0, \quad (1.1)$$

used to model an approximation for surface water waves in a uniform channel [1]. If we note the KdV type equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad (1.2)$$

then we find out the likeness between these equations. Indeed, this similarity is not stochastic. Both of them used to model the waves appear in liquids, compressible fluids, cold plasma and enharmonic crystals which are of surface, hydro-magnetics, acoustic-gravity and acoustic types, respectively. The interesting point is that the main difference between equations (1.1) and (1.2) occurs in the case of short waves (Find more information in [1, 3]).

The physical applications and mathematical properties of the BBM equation (1.1) have been motivated many investigations such as obtaining the exact solutions via finite difference discrete process, global attractor and etc.

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In this paper, we find the similarity reduced ODEs as well as resulted similarity solutions of this equation via weak symmetry implementation. Indeed, the organization of the present paper is as follows: Some historical information on the weak symmetry method are given in section 2. In section 3, we follow [10] in order to describe the theory of weak symmetries. Section 4 is devoted to performing this new class of symmetry methods using the invariant surface condition of the BBM equation (which is indeed the non-classical symmetry method) and infinitesimal generators of the classical symmetry algebra of the KdV type equation as the starting points in the weak symmetry method implementation. Finally, we have compared our results with those related papers using the classical symmetry method in order to clarify the advantages and disadvantages of the both strong and weak symmetry methods.

2. Background

Symmetry methods for differential equations, was originally developed by S. LIE [7]. These methods without any doubt are very useful and algorithmic for analyzing and solving linear and non-linear differential equations. Classification of differential equations as well as linearization of them are some other important applications of the symmetry transformation approach. First G.W. BLUMAN and J.D. COLE introduced the notion of the non-classical symmetry group of differential equations specially for the heat equation in 1969 (Find more information in [2]). For the non-classical method, we seek the invariance of both the original equation and its invariant surface condition, exactly this constraint (i.e invariance surface condition) causes the non-classical solutions which are more general than the classical ones. There are various implementations for performing the non-classical symmetry method, for example, using the compatibility condition has been suggested by G. CAI and X. LING [5].

First the weak symmetries have been introduced by P.J. OLVER, and P. ROSENAU in 1986 as a generalization of the non-classical symmetries with motivation of finding every solutions of the given system. In principle, not only the invariant solutions corresponding to arbitrary transformation groups can be found by the reduction method, but also every possible solution of the system can be found by using some transformation groups. In other words, there are no conditions that need to be placed on the transformation group in order to apply the basic reduction procedure (Find more information in [10]). In the next section, we have an attempt to explain the notation and implementation of the weak symmetry method by considering the BBM equation as an example in order to prepare an appropriate setting.

3. On the weak symmetry method

Symmetry groups of a system of partial differential equations can be defined in two types (see [10]).

Definition. Let Δ be a system of partial differential equations. A strong symmetry group of Δ is a group of transformations G on the space of independent and dependent variables which has the following two properties:

- a) The elements of G transform solutions of the system to other solutions of the system.
- b) The G -invariant solutions of the system are found from a reduced system of differential equations involving a fewer number of independent variables than the original system Δ .

Definition. A weak symmetry group of the system Δ is a group of transformations which satisfies the reduction property (b), but no longer transforms solutions to solutions.

Indeed, there are several transformation groups which don't transform solutions of given equations again to solutions, but their differential invariants enable us to reduce them. In continuation we would illustrate the procedure of performing this method. For this purpose, first consider an arbitrary one-parameter transformation group, then substitute its related differential invariants and their derivatives into the original equation, finally, you will encounter with three different possible cases which in continuation have been illustrated for the (BBM) equation using an appropriate one-parameter transformation group.

3.1. Reduced equation has no parametric variables

Consider the one-parameter group

$$(x, t, u) \mapsto (x + \lambda, t + \lambda, u),$$

So, we have the characteristic equation $dx = dt = du/0$. By substituting the resulted differential invariants i.e. $r = x - t$ and $w = u$, into equation (1.1), we have $w_{rrr} + ww_r = 0$. As we see, this equation has no parameter variable.

3.2. Reduced equation isn't incompatible and has parametric variables

Consider the one-parameter group

$$(x, t, u) \mapsto (\lambda x, t, \lambda u).$$

So, the characteristic equation is $dx/x = dt/0 = du/u$. By substituting the resulted differential invariants $r = t$ and $w = u/x$, in equation (1.1), we have $x(w^2 + w_r) + w = 0$, where $w = 0$ is it's solution and this equation has x as the parametric variable.

3.3. Reduced equation is incompatible and has parametric variables

Consider the one-parameter group

$$(x, t, u) \mapsto (x + 2\lambda t + \lambda^2, t + \lambda, u + 8\lambda t + 4\lambda^2).$$

By substituting the resulted differential invariants i.e. $r = x - t^2$ and $w = u - 4t^2$, in equation (1.1), we have $ww_r + w_r + (8 - 2w_{rrr} - 2w_r)t + 4w_rt^2$, where this equation has t as the parametric variable and it is incompatible. Indeed, from the coefficient of t^2 we have $w_r = 0$ and from the coefficient of t we have $w_{rrr} + w_r = 4$, this means that these equations are incompatible.

4. Implementation of the weak symmetry method for the BBM equation

Since, the weak symmetry method is based on conjecture, so here, the several ideas of performing this method as well as some of its aspects are presented.

4.1. Non-classical symmetries of the BBM equation

There are several implementations to find the non-classical symmetries. Here, we follow the procedure presented by G. CAI et al. which they obtained the non-classical symmetries of the Burgers-Fisher equation based on the compatibility conditions [4].

Consider the following one-parameter group:

$$\begin{aligned}\tilde{x} &= x + \varepsilon \xi(x, t, u) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon \eta(x, t, u) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon \varphi(x, t, u) + O(\varepsilon^2),\end{aligned}\tag{4.3}$$

Assume that the equation $\Delta_1(x, u^{(n)}) := \text{eq}(1.1)$ is invariant under the transformation group (4.3) with the following invariant surface condition:

$$\Delta_2(x, u^{(n)}) := \eta u_t + \xi u_x - \varphi = 0\tag{4.4}$$

This means that $X^{(4)}\Delta_1|_{\Delta_1=0, \Delta_2=0} = 0$, where

$$X = \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_t + \varphi(x, t, u)\partial_u,$$

is the infinitesimal generator of (4.4), and

$$X^{(4)} = X + \varphi^x \partial_{u_x} + \dots + \varphi^{tttt} \partial_{u_{tttt}},$$

is the fourth prolongation of X , with the coefficients defined as $\varphi^J = D_J Q + \xi u_{Jx} + \eta u_{Jt}$, where $Q = \varphi - \xi u_x - \eta u_t$ is the Lie characteristic and $D_J = \sum_{i=0} u_{Ji} \partial_{u_i}$ is the total derivative w.r.t. J (Find more information in [8, 9])

Without loss of generality in condition (4.4), two cases $\eta = 0$ and $\eta = 1$ must be considered.

Case I $\eta = 1$: In this case we have $u_t = \varphi - \xi u_x$. Substituting this expression in (1.1) we have $D_t(\varphi - \xi u_x) = D_t(u_{xxt} - u_x - uu_x)$, where D_t is total derivative w.r.t. t . By substituting ξu_{xx} in both sides of above, we find

$$\begin{aligned}\varphi^t &= u_{xxtt} - u_{xt} - u_t u_x - uu_{xt} + \xi u_{xx} \\ &= D_{xxt}(u_t) - (u+1)D_x(u_t) + (\xi u_x - \varphi)u_x + \xi u_{xx}, \\ &= D_{xxt}(\varphi - \xi u_x) - (u+1)D_x(\varphi - \xi u_x) + (\xi u_x - \varphi)u_x + \xi u_{xx}, \quad (4.5) \\ &= \varphi^{xxt} - \xi u_{xxxt} - (u+1)\varphi^x + (u+1)\xi u_{xx} + (\xi u_x - \varphi)u_x + \xi u_{xx},\end{aligned}$$

By virtue of $D_x(u_t) = D_x(u_{xxt} - u_x - uu_x)$, we have $u_{xt} = u_{xxxt} - u_{xx} - uu_{xx} - u_x^2$. Finally, we find the following governing equation:

$$\varphi^t = \varphi^{xxt} - (u+1)\varphi^x - \varphi u_x, \quad (4.6)$$

where $\varphi^t = D_t(\varphi - \xi u_x) + \xi u_{xt}$, $\varphi^x = D_x(\varphi - \xi u_x) + \xi u_{xx}$, and

$$\varphi^{xxt} = D_{xxt}(\varphi - \xi u_x) + \xi u_{xxxt}.$$

By substituting the coefficient functions $\varphi^t, \varphi^x, \varphi^{xxt}$ into invariance condition (4.6), we are left with a polynomial equation involving the various derivatives of $u(x, t)$ whose coefficients are certain derivatives of ξ and φ . Since, ξ and φ depend only on x, t, u we can equate the individual coefficients to zero, leading to these complete set of determining equations: $\xi_x = \xi_t = \xi_u = 0$, $\varphi = 0$. So, we have $\xi = c_1$, $\varphi = 0$. So, we find the infinitesimal generators of the non-classical symmetries using the above results as follows, when $c_1 = 1$, we have $\sigma_1 = u_x + u_t$, and for $c_1 \neq 0$ the symmetries are $\sigma_2 = u_x$, $\sigma_3 = u_t$. As a result we can state the following proposition:

Proposition. *The non-classical symmetries of the BBM equation in the case of $\eta = 1$, spanned by*

$$\sigma_1 = u_x + u_t, \quad \sigma_2 = u_x, \quad \sigma_3 = u_t. \quad (4.7)$$

As a result of above proposition we have the following group-invariant solutions:

- 1) For $\sigma_1 = u_x + u_t$, substituting it into $\sigma_1(u)$ we find $u = F(x - t)$, where F must satisfy in: $FF' - F''' = 0$
- 2) For $\sigma_2 = u_x$, substituting it into $\sigma_2(u) = 0$ we find $u = F(t)$ for an arbitrary F , so from equation (1.1) we obtain: $u = 0$.
- 3) For $\sigma_3 = u_t$, substituting it into $\sigma_3(u) = 0$ we find $u = F(x)$, where from equation (1.1) F satisfies this equation: $F' + FF' + F''' = 0$.

Case II $\eta = 0$: In this case, without lose of generality we can let $\xi = 1$, so we have: $u_x = \varphi$. Using this we can deduce $A(x, t, u) = \varphi_{xt} - \varphi - u\varphi$. Subsisting this in the determining equation $A\varphi_u + \varphi_t - A_u\varphi - A_x = 0$, we obtain:

$$\varphi_{xt}\varphi_u - 2\varphi\varphi_u - u\varphi\varphi_u + \varphi_t = \varphi_{xtu}\varphi + u\varphi_u\varphi + 2\varphi^2 + \varphi_{xxt} + \varphi_x + u\varphi_x. \quad (4.8)$$

By assuming $\varphi = \varphi(x, t)$ above equation changes into

$$\varphi_t - 2\varphi^2 - \varphi_{xxt} - \varphi_x - u\varphi_x = 0.$$

So we have: $\varphi = 1/(c-2x)$. As a result, we deduce that $u(x, t) = x/(c-2t) + F(t)$ (where F is an arbitrary function) is a solution of (1.1).

4.2. Using the classical symmetries of KdV type equation (1.2)

Since the appearance forms of equation (1.1) and (1.2) are similar, we want to try our chance in order to obtain new similarity reduced ODEs for BBM equation through infinitesimal generators of the classical symmetries (CS) of KdV type equation as the starting constraint. For the classical symmetries of the KdV type equation using Lie classical symmetry we have the next theorem (Since, the proof is computational, to keep scope we don't present it here. Find more information in [8, 9]).

Theorem. *If we consider $X = \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_t + \eta(x, t, u)\partial_u$ as the infinitesimal generator of the classical symmetry group of the KdV type equation (1.2), then we have*

$$\eta = c_1t + c_2, \quad \xi = \frac{1}{3}c_1(x + 2t) + c_3t + c_4, \quad \varphi = -\frac{2}{3}c_1u + c_3, \quad (4.9)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Hence the next corollary could be stated:

Corollary. *The classical symmetries of equation (1.2) i.e. KdV type equation, spanned by:*

$$X_1 = (x + 2t)\partial_x + 3t\partial_t - 2u\partial_u, \quad X_2 = \partial_t, \quad X_3 = t\partial_x + \partial_u, \quad X_4 = \partial_x. \quad (4.10)$$

So, we can consider any linear combinations of given vector fields in the above corollary as the starting constraint of the weak symmetry method. In continuation, we will illustrate the weak symmetry method using some linear combination of X_1, X_2, X_3 and X_4 as the starting point.

Example. Consider the one-parameter transformation group with the infinitesimal generator $X_2 + X_3 = t\partial_x + \partial_t + \partial_u$. The characteristic equation is $dx/t = dt = du$. So, we find the differential invariants as $r = t^2 - 2x$, $w = u/t$. By substituting these new variables in the original equation (1.1) we deduce $(2w_r - 2ww_r - w_{rrr})t^2 = 2w_rt + 4w_{rr} + w$, where t can be considered as the differential parameter. Note that solving the above ODE doesn't give new solution.

Example. Consider the one-parameter transformation group with the infinitesimal generator $X_3 = t\partial_x + \partial_u$. The characteristic equation is $dx/t = dt/0 = du$. So, we can obtain the differential invariants as $r = t$, $w = u - x/t$. By substituting these new variables in the original equation (1.1) we find: $rw_r + w - 1 = 0$, solving this reduced equation we obtain $w = r/(r + c)$. So we can find $u = (tx + x^2 + cx)/(t(x + c))$ as the solution of equation (1.1).

4.3. Some other suggestions

Some other ideas may be useful to reach other solutions of the BBM equation. For example, non-classical potential symmetry method or using classical and non-classical symmetries of other equations which have the similar forms as the BBM equation. Meanwhile, Physical knowledge of the model framework can be so effective in order to reach favorite solutions via weak symmetries. For example if you know your desired solution may be invariant under some scale of specific variables then the weak symmetry method can be started with an appropriate scaling transformation. Since the main goal of this paper was introducing weak symmetry method for BBM equation, we lay away performing of above approaches.

5. More discussions

Now, we want to compare our results with other related papers. Paper [6] is concentrated on the classical symmetries and optimal Lie system of the BBM equation. Comparing with [6], we deduce that in this paper by applying the weak symmetry method we have obtained more similarity solutions and other useful suggestions are presented in order to reach more other solutions.

Taking into account the sections 2 and 3 of [6], the next theorem can be resulted (Find more information in ([8], Chapter 3).

Theorem. *If $u = f(x, t)$ is solution of the BBM equation (1.1), so are the functions*

$$u = f(x - \varepsilon, t), \quad u = f(x - \alpha\varepsilon, t - \varepsilon), \quad u = e^{(u+1)\varepsilon} f(x - \alpha\varepsilon, e^{-\varepsilon}t),$$

where $\varepsilon \ll 1$ and α are arbitrary constants.

Indeed, above theorem characterizes the invariant solutions of the BBM equation, for instance if $u = c$ is a solution of equation (1.1), then from this theorem we obtain $u = ce^{\varepsilon(u+1)}$ as a solution of the BBM equation. For another example, if we consider the solution $u = (tx + x^2 + cx)/(t(x + c))$ of equation (1.1), from this theorem we deduce that

$$u = \frac{(t - \varepsilon)(x - \alpha\varepsilon) + (x - \alpha\varepsilon)^2 + c(x - \alpha\varepsilon)}{(t - \varepsilon)(x - \alpha\varepsilon + c)},$$

(where $\varepsilon \ll 1$ and α, c are arbitrary constants), is again a solution of BBM equation. By using such approach, we are enable to obtain more new solutions for the BBM equation.

Conclusions

In this paper, we have presented a comprehensive explanation of the weak symmetry method as the generalization of the classical Lie symmetry method. Indeed, we have performed the weak symmetry method for the BBM equation which has been fulfilled by applying the non-classical symmetries of the BBM equation and using the classical symmetries of the KdV type equation as the starting constraints. Also, the similarity reduced equations as well as some exact solutions of the BBM equation are obtained via this method. Finally, we have compared our results with papers using the classical symmetry method. Other suggestions for finding new exact solutions are also presented.

References

- [1] BENJAMIN, T.B. and BONA, J.L. and MAHONY, J.J., *Model equations for long waves in nonlinear dispersive systems* Phil. Trans. R. Soc. 1972, 272 4778.
- [2] BLUMAN, G.W. and COLE, J.D., *The general similarity solution of the heat equation*, J. Math. Mech. 18 A969X 1025-1042.
- [3] BONA, J.L. and BRYANT, P.J., *A mathematical model for long waves generated by wave makers in nonlinear dispersive systems* Proc. Cambridge Phil. Soc. 1973, 73 391405.
- [4] CAI, G. and WANG, Y. and ZHANG, F., *Nonclassical symmetries and group invariant solutions of Burgers-Fisher equations*. World Journal of Modelling and Simulation Vol. 3 (2007) No. 4, pp. 305-309.
- [5] CAI, G. and LING, X., *Nonclassical symmetries of a class of nonlinear partial differential equations and compatibility*. World Journal of Modelling and Simulation Vol. 3. 2007. No. 1, pp. 51-57.
- [6] KARACA, M.A. and Hizel, E., *Similarity reductions of Benjamin-Bona-Mahony equation*, Applied Mathematical Sciences, Vol. 2, 2008, no. 10, 463 - 469
- [7] LIE, S., *Über die integration durch bestimmte integrale von einer klasse linear partieller differentialgleichung*, Arch, for Math. 6 A881), pp. 328-368; also Gesammelte Abhandlungen, vol. 3, B.G. Teubner, Leipzig, 1922, pp. 492-523.
- [8] OLVER, P.J., *Applications of Lie groups to differential equations*, New York, Springer, 1986.
- [9] OLVER, P.J., *Equivalence, invariants and symmetry*, Cambridge University Press, 1995.
- [10] OLVER, P.J. and ROSENAU, P., *Group-invariant solutions of differential equations* Siam J., Applied Mathematics, Vol 47, No 2, April 1987.